

On divisors of a quadratic form

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Abstract. An approximation is given for the number of divisors of the quadratic form $n^2 + m^2 + t^2$.

Keywords: Approximation, divisor function, quadratic form.

1. Introduction.

Let $\tau(n)$ denote the number of divisors of n . Hooley [2] studied the behavior of the sum

$$S(x) = \sum_{n \leq x} \tau(n^2 + a)$$

where a is a fixed non zero integer such that $-a$ is not a perfect square. Gafurov [3,4] proved an asymptotic formula for the sum

$$S(x) = \sum_{1 \leq n, m \leq x} \tau(n^2 + m^2).$$

The main purpose of this paper is to obtain an approximation for the sum

$$S(x) = \sum_{1 \leq n, m, t \leq x} \tau(n^2 + m^2 + t^2). \quad (1.1)$$

From definition of $\tau(n)$, we can write (1.1) as

$$\begin{aligned} S(x) = & 2 \sum_{d \leq x\sqrt{3}} \sum_{\substack{1 \leq n, m, t \leq x \\ n^2 + m^2 + t^2 \equiv 0 \pmod{d}}} 1 - \sum_{d \leq x\sqrt{3}} \sum_{\substack{1 \leq n, m, t \leq x \\ n^2 + m^2 + t^2 \equiv 0 \pmod{d} \\ n^2 + m^2 + t^2 \leq dx\sqrt{3}}} 1 \\ & = 2S_1 - S_2. \end{aligned} \quad (1.2)$$

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Let $e(s)$ denote the exponential function, $e(s) = e^{2\pi i s}$, and $\zeta(s)$ the Riemann Zeta Function. Let γ be Euler's constant. Let $\rho(n)$ denote the number of solutions of the congruence

$$x^2 + y^2 + z^2 \equiv 0 \pmod{n}, \quad 1 \leq x, y, z \leq n. \quad (1.3)$$

Our first result is an asymptotic formula for the function $\rho(n)$.

Theorem 1. *The following formula holds*

$$\sum_{n \leq x} \rho(n) = \frac{4}{15} \frac{\zeta(3)}{\zeta(4)} x^3 + O(x^2 \log x). \quad (1.4)$$

Using Theorem 1 and (1.2) we shall get the following approximation for the sum (1.1).

Theorem 2. *The divisor function of a quadratic form verifies the following formulas*

$$\begin{aligned} & \left(2C - \frac{222\sqrt{3}}{25} \frac{\zeta(3)}{\zeta(4)} \right) x^3 + O(x^2 \log x) \\ & < \sum_{1 \leq n, m, t \leq x} \tau(n^2 + m^2 + t^2) - \frac{8}{5} \frac{\zeta(3)}{\zeta(4)} x^3 \log x \\ & \leq \left(2B - \frac{2}{225\sqrt{3}} \frac{\zeta(3)}{\zeta(4)} \right) x^3 + O(x^2 \log x) \end{aligned} \quad (1.5)$$

where

$$B = A + \frac{\zeta(3)}{\zeta(4)} \frac{2}{5} (\log 3 + 9 + 8\sqrt{3}), \quad C = A + \frac{\zeta(3)}{\zeta(4)} \frac{2}{5} (\log 3 + 9 - 8\sqrt{3})$$

for

$$A = \frac{4}{5} \frac{\zeta(3)}{\zeta(4)} \gamma + 2 \left(\frac{\zeta(s-1)}{\zeta(s)} \right)'_{s=4} - 6 \left(\frac{1}{2^s - 1} \frac{\zeta(s-1)}{\zeta(s)} \right)'_{s=4}. \quad (1.6)$$

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2. Preliminary results

Expressing the number of solutions of the congruence (1.3) by Gauss Sums we obtain the following lemma.

Lemma 1. *Let $\rho(p^\alpha)$ the number of solutions of the congruence $x^2 + y^2 + z^2 \equiv 0 \pmod{p^\alpha}$, $1 \leq x, y, z \leq p^\alpha$. Then we have*

(a) when p is an odd prime

$$\rho(p^\alpha) = \begin{cases} p^{2\alpha} + p^{2\alpha-1} - p^{3\alpha/2-1}, & \text{if } \alpha \text{ is even} \\ p^{2\alpha} + p^{2\alpha-1} - p^{(3\alpha-1)/2}, & \text{if } \alpha \text{ is odd,} \end{cases}$$

(b) when $p = 2$

$$\rho(2^\alpha) = \begin{cases} 2^{3\alpha/2}, & \text{if } \alpha \text{ is even} \\ 2^{(3\alpha+1)/2}, & \text{if } \alpha \text{ is odd.} \end{cases}$$

Proof. (a) We can express the number of solutions of the congruence in the form

$$\rho(p^\alpha) = \frac{1}{p^\alpha} \sum_{a=1}^{p^\alpha} \left(\sum_{u=1}^{p^\alpha} e \left(\frac{au^2}{p^\alpha} \right) \right)^3 \quad (2.1)$$

Moreover, (see [6], Chapter 1, §3) if p is an odd prime number and $p \nmid a$,

$$S(a, p^\gamma) = \sum_{u=1}^{p^\gamma} e \left(\frac{au^2}{p^\gamma} \right) = \begin{cases} \left(\frac{a}{p} \right) i^{((p-1)/2)^2} p^{\gamma/2}, & \text{if } \gamma \equiv 1 \pmod{2} \\ p^{\gamma/2}, & \text{if } \gamma \equiv 0 \pmod{2} \end{cases} \quad (2.2)$$

where $\left(\frac{a}{p} \right)$ is Legendre's Symbol. Therefore, we can write

$$\rho(p^\alpha) = \frac{1}{p^\alpha} \sum_{a=1, p \nmid a}^{p^\alpha} (S(a, p^\alpha))^3 + \frac{1}{p^\alpha} \sum_{\beta=1}^{\alpha} \sum_{a=1, p \nmid a}^{p^{\alpha-\beta}} p^{3\beta} (S(a, p^{\alpha-\beta}))^3.$$

First, we suppose $\alpha \equiv 0 \pmod{2}$. We have

$$\begin{aligned} \rho(p^\alpha) &= \frac{1}{p^\alpha} \sum_{a=1, p \nmid a}^{p^\alpha} p^{3\alpha/2} + \frac{1}{p^\alpha} \sum_{\gamma=1}^{\alpha/2} \sum_{a=1, p \nmid a}^{p^{\alpha-2\gamma}} p^{6\gamma} p^{3(\alpha-2\gamma)/2} + \\ &+ \frac{1}{p^\alpha} \sum_{\gamma=1}^{\alpha/2} \sum_{a=1, p \nmid a}^{p^{\alpha-2\gamma+1}} p^{6\gamma-3} p^{3(\alpha-2\gamma+1)/2} \left(\frac{a}{p} \right)^3 i^{3((p-1)/2)^2}. \end{aligned}$$

As $\sum_{a=1, p \nmid a}^p \left(\frac{a}{p}\right) = 0$, we obtain

$$\rho(p^\alpha) = p^{2\alpha} + p^{2\alpha-1} - p^{3\alpha/2-1}. \quad (2.3)$$

Analogously, for $\alpha \equiv 1 \pmod{2}$, we obtain

$$\rho(p^\alpha) = p^{2\alpha} + p^{2\alpha-1} - p^{(3\alpha-1)/2}. \quad (2.3')$$

(b) In the case $p = 2$, when $2 \nmid a$, the Gauss sum holds

$$S(a, 2^\gamma) = \sum_{x=1}^{2^\gamma} e\left(\frac{ax^2}{2^\gamma}\right) = \begin{cases} 1, & \text{if } \gamma = 0 \\ 0, & \text{if } \gamma = 1 \\ 2^{\gamma/2}(1+i^a), & \text{if } \gamma > 0 \text{ and even} \\ 2^{\gamma/2}(1+i)e^{(i\pi/4)(a-1)}, & \text{if } \gamma > 1 \text{ and odd} \end{cases}$$

(see [5] Thm. 3, Chapter 11). Then

$$\rho(2^\alpha) = \frac{1}{2^\alpha} \sum_{a=1, 2 \nmid a}^{2^\alpha} (S(a, 2^\alpha))^3 + \frac{1}{2^\alpha} \sum_{\beta=1}^{\alpha} \sum_{a=1, 2 \nmid a}^{2^{\alpha-\beta}} 2^{3\beta} (S(a, 2^{\alpha-\beta}))^3.$$

When $\alpha \equiv 0 \pmod{2}$, $\alpha > 0$, we have

$$\begin{aligned} \rho(2^\alpha) &= \frac{1}{2^\alpha} \sum_{a=1, 2 \nmid a}^{2^\alpha} 2^{\frac{3\alpha}{2}} (1+i^a)^3 + \frac{1}{2^\alpha} \sum_{\gamma=1}^{\frac{\alpha}{2}-1} \sum_{a=1, 2 \nmid a}^{2^{\alpha-2\gamma}} 2^{6\gamma} 2^{\frac{3}{2}(\alpha-2\gamma)} (1+i^a)^3 \\ &\quad + \frac{(1+i)^3}{2^\alpha} \sum_{\gamma=1}^{\alpha/2-1} \sum_{a=1, 2 \nmid a}^{2^{\alpha-2\gamma+1}} 2^{6\gamma-3} 2^{\frac{3}{2}(\alpha-2\gamma+1)} e^{\frac{3i\pi}{4}(a-1)} + 2^{2\alpha}. \end{aligned}$$

It says

$$\rho(2^\alpha) = 2^{3\alpha/2}. \quad (2.4)$$

In same form, when $\alpha \equiv 1 \pmod{2}$, we deduce

$$\rho(2^\alpha) = 2^{(3\alpha+1)/2}. \quad (2.4')$$

□

Let $F(s)$ be the Dirichlet function of $\rho(n)$,

$$F(s) = \sum_{n=1}^{\infty} \frac{\rho(n)}{n^s}.$$

By Lemma 1 and the properties of Dirichlet series we have

$$F(s) = \zeta(s-2)\zeta(2s-3) \sum_{n=1}^{\infty} \frac{\mu(n)f(n)}{n^{2s-2}}$$

where $\mu(n)$ is the Möbius function and $f(n)$ is the multiplicative function

$$f(n) = \begin{cases} 1, & \text{if } n \text{ is odd} \\ 4, & \text{if } n \text{ is even.} \end{cases}$$

Hence, we can express

$$\rho(n) = n^2 \sum_{d^2|n} \frac{g(d)}{d^2} \quad (2.5)$$

with

$$g(d) = \begin{cases} \varphi(d), & \text{if } d \text{ is odd} \\ -2\varphi(d), & \text{if } d \text{ is even} \end{cases} \quad (2.6)$$

where $\varphi(d)$ is Euler's function,

$$\varphi(d) = \#\{n \in \mathbb{N} \mid 1 \leq n \leq d, (n, d) = 1\}.$$

It is known that

$$\sum_{\substack{n=1 \\ (2,n)=1}}^{\infty} \frac{\varphi(n)}{n^s} = \prod_{p \neq 2} \left\{ \sum_{\alpha=0}^{\infty} \frac{\varphi(p^\alpha)}{p^{\alpha s}} \right\} = \frac{2^s - 2}{2^s - 1} \frac{\zeta(s-1)}{\zeta(s)}, \quad s > 2$$

and (see Thm. 11.12 of [1])

$$\sum_{\substack{n=1 \\ (2,n)=1}}^{\infty} \frac{\varphi(n) \log n}{n^4} = - \left(\frac{\zeta(s-1)}{\zeta(s)} \right)'_{s=4} + \left(\frac{1}{2^s - 1} \frac{\zeta(s-1)}{\zeta(s)} \right)'_{s=4}, \quad s > 2.$$

Lemma 2. *It holds*

$$\sum_{n \leq x} \frac{g(n)}{n^4} = \frac{4}{5} \frac{\zeta(3)}{\zeta(4)} + O\left(\frac{\log x}{x^3}\right). \quad (2.7)$$

Proof. From (2.6) this sum is

$$\sum_{n \leq x} \frac{g(n)}{n^4} = 3 \sum_{\substack{n \leq x \\ (2,n)=1}} \frac{\varphi(n)}{n^4} - 2 \sum_{n \leq x} \frac{\varphi(n)}{n^4}. \quad (2.8)$$

Notice that, from Abel summation Formula

$$\sum_{n \leq x} \frac{\varphi(n)}{n^4} = \frac{\zeta(3)}{\zeta(4)} - \frac{1}{2\zeta(2)x^2} + O\left(\frac{\log x}{x^3}\right). \quad (2.9)$$

As $\varphi(n) = n \sum_{d|n} \frac{\mu(d)}{d}$, we can write

$$\sum_{\substack{n \leq x \\ (2,n)=1}} \frac{\varphi(n)}{n^4} = \sum_{\substack{n \leq x \\ (2,n)=1}} \frac{\mu(n)}{n^4} \sum_{\substack{m \leq x/n \\ (2,m)=1}} \frac{1}{m^3}. \quad (2.10)$$

We need the following relations

$$\sum_{\substack{n \leq x \\ (2,n)=1}} \frac{1}{n^3} = \frac{7}{8}\zeta(3) - \frac{1}{4}x^{-2} + O(x^{-3}) \quad (2.11)$$

and

$$\sum_{\substack{n \leq x \\ (q,n)=1}} \frac{\mu(n)}{n^\alpha} = \frac{q^\alpha}{\zeta(\alpha)J_\alpha(q)} + O\left(\frac{\sigma_{-1+\epsilon}^*(q)\delta(x)}{x^{\alpha-1}}\right) \quad (2.12)$$

where $J_\alpha(n) = \sum_{dq=n} \mu(d)q^\alpha$, where $\sigma_s^*(q)$ is the sum of s -th powers of the square-free divisors of q , and where $\delta(x) = \exp\{-A \log^{\frac{3}{5}} x (\log \log x)^{-\frac{1}{5}}\}$, for A a positive constant (see [7], Lemma 3.6). From (2.10), (2.11) and (2.12) we obtain

$$\sum_{\substack{n \leq x \\ (2,n)=1}} \frac{\varphi(n)}{n^4} = \frac{14}{15} \frac{\zeta(3)}{\zeta(4)} - \frac{1}{3\zeta(2)x^2} + O\left(\frac{\log x}{x^3}\right). \quad (2.13)$$

Then, replacing (2.9) and (2.13) in (2.8) we obtain (2.7). $\square$

3. Proofs of Theorems

Proof of Theorem 1. By (2.5), one has

$$\begin{aligned} \sum_{n \leq x} \rho(n) &= \sum_{n \leq \sqrt{x}} n^2 g(n) \sum_{\substack{m \leq \frac{x}{n^2}}} m^2 \\ &= \frac{x^3}{3} \sum_{n \leq \sqrt{x}} \frac{g(n)}{n^4} + O\left(x^2 \sum_{n \leq \sqrt{x}} \frac{|g(n)|}{n^2}\right). \end{aligned}$$

Notice that $\sum_{n \leq x} \frac{g(n)}{n^2} = O(\log x)$. From this bound and Lemma 2 we have

$$\sum_{n \leq x} \rho(n) = \frac{x^3}{3} \left\{ \frac{4}{5} \frac{\zeta(3)}{\zeta(4)} + O\left(\frac{\log x}{x^{3/2}}\right) \right\} + O(x^2 \log x)$$

and (1.4) is proved. $\square$

Proof of Theorem 2. From (1.2) we can write

$$\sum_{1 \leq n, m, t \leq x} \tau(n^2 + m^2 + t^2) = 2S_1 - S_2, \quad (3.1)$$

where

$$S_2 = \sum_{d \leq x\sqrt{3}} \sum_{\substack{1 \leq n, m, t \leq x \\ n^2 + m^2 + t^2 \equiv 0 \pmod{d} \\ n^2 + m^2 + t^2 \leq dx\sqrt{3}}} 1 = \sum_{d \leq x\sqrt{3}} S(d).$$

Now, the inner sum $S(d)$ is bounded by the number of integer points (n, m, t) such that $1 \leq n, m, t \leq \sqrt{dx\sqrt{3} - 2}$ and $n^2 + m^2 + t^2 \equiv 0 \pmod{d}$.

For x any real number, let $[x]$ denote the largest integer $\leq x$. Making the following change of variables, $n = n_1d + v_1$, $m = m_1d + v_2$, $t = t_1d + v_3$, $1 \leq v_1, v_2, v_3 \leq d$, we have

$$S_2 \leq \sum_{d \leq x\sqrt{3}} \sum_{1 \leq v_1, v_2, v_3 \leq d} \theta(v_1)\theta(v_2)\theta(v_3)$$

where we denote

$$\theta(v_i) = \left[\left(\sqrt{dx\sqrt{3} - 2} - v_i \right) / d \right] + 1$$

for every $i = 1, 2, 3$.

On the other hand, $S(d)$ is bigger than the number of integer points (n, m, t) such that $1 \leq n, m, t \leq \sqrt{dx/\sqrt{3} - 2}$ and $n^2 + m^2 + t^2 \equiv 0 \pmod{d}$. So

$$S_2 \geq \sum_{d \leq x/\sqrt{3}} \sum_{1 \leq v_1, v_2, v_3 \leq d} \Phi(v_1)\Phi(v_2)\Phi(v_3)$$

where

$$\Phi(v_i) = \left[\left(\sqrt{(dx/\sqrt{3}) - 2} - v_i \right) / d \right] + 1$$

for every $i = 1, 2, 3$.

Using the properties of $[x]$, we obtain

$$\sum_{d \leq x/\sqrt{3}} \rho(d) \left[\frac{y_1}{d} \right]^3 < S_2 \leq \sum_{d \leq x\sqrt{3}} \rho(d) \left(\left[\frac{y}{d} \right] + 1 \right)^3 \quad (3.2)$$

where $y = \sqrt{dx\sqrt{3} - 2}$ and $y_1 = \sqrt{\frac{dx}{\sqrt{3}}} - 2$. Then we can write

$$\begin{aligned} S_2 &\leq 3^{3/4} x^{3/2} \sum_{d \leq x\sqrt{3}} \frac{\rho(d)}{d^{3/2}} + 3^{3/2} x \sum_{d \leq x\sqrt{3}} \frac{\rho(d)}{d} \\ &\quad + 3^{5/4} x^{1/2} \sum_{d \leq x\sqrt{3}} \frac{\rho(d)}{d^{1/2}} + \sum_{d \leq x\sqrt{3}} \rho(d). \end{aligned} \quad (3.3)$$

Using Abel summation formula and Theorem 1, we get

$$\sum_{n \leq x} \frac{\rho(n)}{n} = \frac{1}{x} A(x) + \int_1^x \frac{A(t)}{t^2} dt + O(x) = \frac{2}{5} \frac{\zeta(3)}{\zeta(4)} x^2 + O(x \log x), \quad (3.4)$$

where $A(t) = \sum_{n \leq t} \rho(n)$. In the same form we deduce

$$\sum_{n \leq x} \frac{\rho(n)}{\sqrt{n}} = \frac{8}{25} \frac{\zeta(3)}{\zeta(4)} x^{5/2} + O(x^{3/2} \log x) \quad (3.5)$$

and

$$\sum_{n \leq x} \frac{\rho(n)}{n^{3/2}} = \frac{8}{15} \frac{\zeta(3)}{\zeta(4)} x^{3/2} + O(x^{1/2} \log x). \quad (3.6)$$

Therefore, from (3.3), (3.4)-(3.6) and Theorem 1 we deduce

$$S_2 \leq \frac{222\sqrt{3}}{25} \frac{\zeta(3)}{\zeta(4)} x^3 + O(x^2 \log x). \quad (3.7)$$

Analogously, from (3.2) we have

$$\begin{aligned} S_2 &> 3^{-3/4} x^{3/2} \sum_{d \leq x/\sqrt{3}} \frac{\rho(d)}{d^{3/2}} - 3^{1/2} x \sum_{d \leq x/\sqrt{3}} \frac{\rho(d)}{d} \\ &\quad + 3^{3/4} x^{1/2} \sum_{d \leq x/\sqrt{3}} \frac{\rho(d)}{d^{1/2}} - \sum_{d \leq x/\sqrt{3}} \rho(d) + O(x^2). \end{aligned}$$

Then, from (3.4), (3.5), (3.6) and Theorem 1, we have

$$S_2 > \frac{2}{225\sqrt{3}} \frac{\zeta(3)}{\zeta(4)} x^3 + O(x^2 \log x). \quad (3.8)$$

For the first sum, S_1 , it has

$$\sum_{d \leq x\sqrt{3}} \rho(d) \left[\frac{x}{d} \right]^3 < S_1 \leq \sum_{d \leq x\sqrt{3}} \rho(d) \left(\left[\frac{x}{d} \right] + 1 \right)^3$$

As before, by Theorem 1 and partial summation, we obtain

$$\sum_{n \leq x} \frac{\rho(n)}{n^2} = \frac{4}{5} \frac{\zeta(3)}{\zeta(4)} x + O(\log x). \quad (3.9)$$

In a similar manner to the proof of Theorem 1, one has from (2.5)

$$\sum_{n \leq x} \frac{\rho(n)}{n^3} = \sum_{n \leq \sqrt{x}} \frac{g(n)}{n^4} \sum_{m \leq x/n^2} \frac{1}{m}$$

and from Thm. 3.2 of [1]

$$\begin{aligned} \sum_{n \leq x} \frac{\rho(n)}{n^3} &= \sum_{n \leq \sqrt{x}} \frac{g(n)}{n^4} \left\{ \log\left(\frac{x}{n^2}\right) + \gamma + O\left(\frac{n^2}{x}\right) \right\} \\ &= (\log x + \gamma) \sum_{n \leq \sqrt{x}} \frac{g(n)}{n^4} - 2 \sum_{n \leq \sqrt{x}} \frac{g(n) \log n}{n^4} + O\left(\frac{\log x}{x}\right). \end{aligned}$$

As in (2.8) and (2.9)

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{g(n) \log n}{n^s} &= 3 \sum_{\substack{n=1 \\ (n,2)=1}}^{\infty} \frac{\varphi(n) \log n}{n^s} - 2 \sum_{n=1}^{\infty} \frac{\varphi(n) \log n}{n^s}, \quad s > 2 \\ \sum_{n \leq \sqrt{x}} \frac{g(n) \log n}{n^4} &= 3 \left(\frac{1}{2^s - 1} \frac{\zeta(s-1)}{\zeta(s)} \right)'_{s=4} - \left(\frac{\zeta(s-1)}{\zeta(s)} \right)'_{s=4} + O\left(\frac{\log x}{x}\right). \end{aligned}$$

So

$$\sum_{n \leq x} \frac{\rho(n)}{n^3} = \frac{4}{5} \frac{\zeta(3)}{\zeta(4)} \log x + A + O\left(\frac{\log x}{x}\right), \quad (3.10)$$

where A is the constant (1.6).

From (1.4), (3.4), (3.9) and (3.10), in analogy to (3.7) and (3.8), we obtain that

$$S_1 \leq \frac{4}{5} \frac{\zeta(3)}{\zeta(4)} x^3 \log x + Bx^3 + O(x^2 \log x) \quad (3.11)$$

where

$$B = A + \frac{\zeta(3)}{\zeta(4)} \frac{2}{5} (\log 3 + 9 + 8\sqrt{3}),$$

and

$$S_1 > \frac{4}{5} \frac{\zeta(3)}{\zeta(4)} x^3 \log x + Cx^3 + O(x^2 \log x) \quad (3.12)$$

where

$$C = A + \frac{\zeta(3)}{\zeta(4)} \frac{2}{5} (\log 3 + 9 - 8\sqrt{3}).$$

From (3.1), (3.7) and (3.12) we get the following inequality

$$\begin{aligned} \sum_{1 \leq n, m, t \leq x} \tau(n^2 + m^2 + t^2) &> \\ &> \frac{8}{5} \frac{\zeta(3)}{\zeta(4)} x^3 \log x + \left(2C - \frac{222\sqrt{3}}{25} \frac{\zeta(3)}{\zeta(4)} \right) x^3 + O(x^2 \log x). \end{aligned} \quad (3.13)$$

And from (3.1), (3.8) and (3.11) we get

$$\begin{aligned} \sum_{1 \leq n, m, t \leq x} \tau(n^2 + m^2 + t^2) &\leq \\ &\leq \frac{8}{5} \frac{\zeta(3)}{\zeta(4)} x^3 \log x + \left(2B - \frac{2}{225\sqrt{3}} \frac{\zeta(3)}{\zeta(4)} \right) x^3 + O(x^2 \log x). \end{aligned} \quad (3.14)$$

The Theorem 2 follows from inequalities (3.13) and (3.14). $\square$

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